
A New Algorithm for Stand Table Projection Models

Quang V. Cao and V. Clark Baldwin, Jr.

ABSTRACT. The constrained least squares method is proposed as an algorithm for projecting stand tables through time. This method consists of three steps: (1) predict survival in each diameter class, (2) predict diameter growth, and (3) use the least squares approach to adjust the stand table to satisfy the constraints of future survival, average diameter and stand basal area. The new method was evaluated against the Weibull parameter-recovery approach and another stand table projection method, using data from direct-seeded stands of loblolly pines (*Pinus taeda* L.). The constrained least squares method provided the best goodness-of-fit statistics (K-S, χ^2 , and an error index) that were significantly different from those produced by the other two methods. This new algorithm can be employed in cases where diameter data do not necessarily follow the Weibull distribution. For. Sci. 45(4):506–511.

Additional Key Words: *Pinus taeda*, loblolly pine, direct-seeded stands, Weibull distribution, constrained least squares.

STAND TABLES GIVE NUMBER OF TREES in each diameter class. They supply information on stand structure for calculation of product volumes and therefore play an important role in growth and yield modeling. A stand table projection model provides prediction of a future stand table based on the current stand table. The simplest stand table projection models apply tree mortality information and diameter growth rates in adjusting the stand table (Chapman and Meyer 1949, Avery and Burkhart 1994).

Clutter and Jones (1980) and Pienaar and Harrison (1988) calculated number of surviving trees for each diameter class based on relative size (tree basal area over average basal area per tree) and then projected diameter growth for each diameter class.

Nepal and Somers (1992) projected the stand table using a diameter growth equation implied from the Weibull distribution (Bailey 1980), then adjusted the stand table such that basal area and trees per unit area matched either observed or predicted values. Their model performed better than Pienaar and Harrison's (1988) model in projecting stand tables of natural even-aged longleaf pine (*Pinus palustris* Mill.) stands. Tang et al. (1997) derived the relationship between diameter cumulative distributions and stand-level attributes at two stand ages. The parameters of the tree survival and diameter growth func-

tions were then recovered from future average diameter, quadratic mean diameter, and survival.

This article introduces a new algorithm for stand table projection models. The resulting future stand table produces estimates of stand basal area, number of trees per hectare, and average tree diameter that are compatible with either observed values or predicted values from growth and yield models.

Data

Data were collected from 148 permanent plots in loblolly pine (*Pinus taeda* L.) direct-seeded stands in Louisiana. These fixed-radius plots varied in size from 0.021 to 0.048 ha. The plots were established between 1960 and 1971 in Rapides Parish (107 plots), Natchitoches Parish (27 plots), and Union Parish (14 plots), and were measured from 2 to 6 times (Table 1). Table 2 shows the distribution of 679 observations by stand age and basal area. These observations constituted 527 growth periods ranging from 2 to 5 yr.

Methods

The new algorithm for stand table projection consists of three steps: (1) computing survival and allocating mortality; (2) deriving diameter growth for individual trees; and (3)

Quang V. Cao, School of Forestry, Wildlife, and Fisheries, Louisiana Agricultural Experiment Station, Louisiana State University Agricultural Center, Baton Rouge, LA 70803—Phone: (225) 388-4218; Fax: (225) 388-4227; E-mail: QCAO@LSU.EDU and V.Clark Baldwin, Jr., USDA Forest Service, Southern Research Station, Pineville, LA 71360.

Manuscript received November 26, 1997. Accepted April 6, 1999.

Copyright © 1999 by the Society of American Foresters

Table 1. Distribution of 148 loblolly pine plots by measurement age.

No. of plots	Measurement ages
51	8, 11, 13, 16, 21
5	8, 11, 16, 21
13	9, 14, 18, 23
3	10, 14, 19
18	10, 14, 19, 24
4	10, 14, 24
2	11, 14, 17
16	11, 14, 17, 22
7	11, 16, 21, 26
18	11, 16, 21, 26, 30, 34
1	14, 18, 23
2	14, 19, 24
8	16, 21, 26, 30, 34

adjusting projected diameters to match future average diameter and stand basal area.

Survival

Since we have no information regarding what went on during the growing period, the alternatives are to assume that all mortality occurs at either the beginning or the end of the period. The drawback of the latter is that dead trees are also included in tracking diameter growth. Therefore we assumed in this model that all mortality occurs at the beginning of the growth period. The surviving number of trees for the i th diameter class was predicted using the following survival function:

$$\hat{n}_{2i} = n_{1i} \{1 - \exp [b_1 - (D_i - D_{min_1} + 1)]\} \quad (1)$$

where

n_{1i} = current number of trees per hectare in the i th diameter class, $i = 1, 2, \dots, p$,

p = number of diameter classes,

\hat{n}_{2i} = future surviving number of trees per hectare in the i th diameter class,

D_i = midpoint diameter of the i th class,

D_{min_1} = midpoint of the current minimum diameter class, and

b_1 = coefficient to be determined.

Based on this equation, more trees survive to the end of the growing period for large diameter classes as compared to small diameter classes, relative to the minimum diameter class. The coefficient b_1 , which is negative, is calculated such that \hat{n}_{2i} will sum up to N_2 , the total future surviving trees per hectare. A numerical method such as Newton-Raphson or secant method (Press et al. 1996) can be used to solve for b_1 .

Since mortality rate is not evenly distributed among diameter classes, the diameter distribution will change after mortality. As a result, the stand attributes need to be updated. The current average diameter (\bar{D}_1) and basal area per hectare (B_1) after mortality are given by:

$$\bar{D}_1 = \frac{\sum_i \hat{n}_{2i} D_i}{N_2} \quad (2)$$

Table 2. Distribution of observations from direct-seeded stands of loblolly pines by stand age and basal area.

Age (yr) from seed	Stand basal area (m ² /ha)						Total
	<5	10	20	30	40	50	
8–10	15	60	18	2			95
11–13	3	35	76	36	1		151
14–16	1	22	28	60	37		148
17–19		12	12	17	12	2	55
20–22		9	13	21	58	6	107
23–25		1	4	15	9	9	38
26–28				3	29	1	33
30				5	17	4	26
34				5	18	3	26
All	19	139	151	164	181	25	679

$$B_1 = K \sum_i \hat{n}_{2i} D_i^2 \quad (3)$$

where $K = \pi/40,000$ (to convert diameter in cm to area in square meters).

Diameter Growth

Increase in Minimum Diameter

For some plots, the minimum diameter increased at the end of the growth period. It was very difficult to model this change, which did not strongly relate to other stand variables. The minimum diameter was assumed to increase (i.e., all trees in the current minimum diameter class either die or move up to higher classes) when there is a sufficient shift in diameter distribution based on the arithmetic and quadratic mean diameters.

Tang et al. (1997) expressed future diameter (X_2) as a function of current diameter (X_1):

$$X_2 = b_0 + b_1 X_1 + \varepsilon \quad (4)$$

where ε is a stochastic error term that contains both random and nonlinear components of the tree diameter growth. The future minimum diameter (D_{min_2}) can be modeled from current minimum diameter (D_{min_1}) using the above diameter growth function without the error term:

$$D_{min_2} = b_0 + b_1 D_{min_1} \quad (5)$$

Tang et al. (1997) showed that the coefficients b_0 and b_1 of (5) may be obtained from

$$b_1^2 = \frac{Q_2^2 - (\bar{D}_2)^2}{Q_1^2 - (\bar{D}_1)^2} \quad (6)$$

and

$$b_0 = \bar{D}_2 - b_1 \bar{D}_1 \quad (7)$$

where Q_j and \bar{D}_j are quadratic mean diameter and average diameter, respectively, at time j .

In this model, the change in minimum diameter, in terms of number of diameter classes, was determined as the largest integer that was less than or equal to $(D_{min_2} - D_{min_1})/h$, where h is the class width. For example, if the increase in minimum diameter for the growth period was 5.1 cm, then the

diameter distribution would be shifted forward (to the right) by two classes or 4 cm if 2 cm diameter classes are used.

Implied Diameter Growth

Procedures used in this step were described in detail by Nepal and Somers (1992). Parameters of a Weibull distribution (Bailey and Dell 1973) were recovered from \bar{D}_1 and B_1 to approximate the current diameter distribution of the stand immediately after mortality. Similarly, parameters of another Weibull distribution to characterize the future diameter distribution were recovered from \bar{D}_2 and B_2 , the future average diameter and basal area per hectare, respectively. The Weibull cumulative distribution function at time i , F_i , having location parameter a_i , scale parameter b_i , and shape parameter c_i is defined as follows:

$$F_i(x) = 1 - \exp \left[- \left(\frac{x - a_i}{b_i} \right)^{c_i} \right] \quad (8)$$

An implied diameter growth function (Bailey 1980) was derived from the Weibull parameters at times 1 and 2:

$$X_2 = a_2 + b_2 \left(\frac{X_1 - a_1}{b_1} \right)^{\frac{c_1}{c_2}} \quad (9)$$

where X_j is tree diameter at time j .

Finally, the movement of trees from one diameter class to another was computed using the assumption that trees in each diameter class followed a doubly-truncated Weibull distribution. Suppose the i th diameter class is specified by the lower and upper limits, l_i and u_i , respectively. If d_1 and d_2 are diameter values in the i th class ($l_i \leq d_1 \leq d_2 \leq u_i$), then number of trees in the interval $[d_1, d_2]$, or $n_{[d_1, d_2]}$, is given by

$$n_{[d_1, d_2]} = n_{1i} \left[\frac{F_i(d_2) - F_i(d_1)}{F_i(u_i) - F_i(l_i)} \right] \quad (10)$$

where n_{1i} is the number of trees in the i th diameter class at time 1.

One item distinguishes the procedures presented in this step from Nepal and Somers' (1992) method (hereafter referred to as the N&S method). The values of \bar{D}_1 and B_1 for the current stand were calculated before mortality in the N&S method, but were computed immediately after mortality in the proposed approach. The benefit of this approach is that diameter growth estimation is based only on trees that survive the growing period.

Adjustment of Stand Table

The result from the previous step was a new stand table for the future stand, with \hat{n}_{2i} being the number of trees per acre in the i th diameter class at time period 2. This new stand table will provide values of average diameter and stand basal area different from \bar{D}_2 and B_2 .

A constrained least squares (LS) procedure, developed by Matney et al. (1990) for their individual tree model, was modified in this study to adjust the future stand table. The final number of trees per hectare in the i th diameter class, n_{2i} , was calculated by minimizing

$$\sum_i (n_{2i} - \hat{n}_{2i})^2 \quad (11)$$

subject to the following constraints:

$$\sum n_{2i} = N_2 \quad (12a)$$

$$\sum n_{2i} D_i = N_2 \bar{D}_2 \quad (12b)$$

$$\sum n_{2i} D_i^2 = B_2 / K \quad (12c)$$

where the summation signs denote the sum over all diameter classes. This step can be interpreted as adjusting values of the stand table (\hat{n}_{2i}) to new values (n_{2i}) such that the three constraints (12a to 12c) are met.

The above constrained least squares problem can be rewritten as:

$$\begin{aligned} \min. & \sum (n_{2i} - \hat{n}_{2i})^2 + 2\lambda_1 (\sum n_{2i} - N_2) \\ & + 2\lambda_2 (\sum n_{2i} D_i - N_2 \bar{D}_2) \\ & + 2\lambda_3 (\sum n_{2i} D_i^2 - B_2 / K) \end{aligned} \quad (13)$$

where Σ denotes the sum over all diameter classes (for values of i from 1 to p), p is the number of diameter classes, and λ_j 's are Lagrangian multipliers.

The solution to this problem is given by differentiating (13) with respect to n_{2i} and then setting the derivative equal to zero:

$$n_{2i} = \hat{n}_{2i} - (\lambda_1 + \lambda_2 D_i + \lambda_3 D_i^2) \quad (14)$$

or

$$\lambda_1 + \lambda_2 D_i + \lambda_3 D_i^2 = \hat{n}_{2i} - n_{2i} \quad (15)$$

The Lagrangian multipliers (λ_j 's) can be easily solved from the following system of three linear equations:

$$\lambda_1 p + \lambda_2 \sum D_i + \lambda_3 \sum D_i^2 = \sum \hat{n}_{2i} - N_2 \quad (16a)$$

$$\lambda_1 \sum D_i + \lambda_2 \sum D_i^2 + \lambda_3 \sum D_i^3 = \sum \hat{n}_{2i} D_i - N_2 \bar{D}_2 \quad (16b)$$

$$\lambda_1 \sum D_i^2 + \lambda_2 \sum D_i^3 + \lambda_3 \sum D_i^4 = \sum \hat{n}_{2i} D_i^2 - B_2 / K \quad (16c)$$

where Σ denotes the sum over all diameter classes (for values of i from 1 to p). Note that equation (16a) is obtained by summing (15) over all diameter classes. Multiplying equation (15) by D_i or D_i^2 and then summing over all diameter classes gives equation (16b) or (16c), respectively.

Sometimes the computed value of n_{2i} was negative for a particular i th diameter class. This often happened when there were "gaps" in the diameter distribution (i.e., when the observed number of trees is zero for that diameter class). If n_{2i} is less than 0, \hat{n}_{2i} was set to zero and the adjustment step was repeated. This procedure was necessary for 21% of the distributions; and just one adjustment was adequate in most cases.

A Numerical Example

The following example presents the results of a stand table projection from age 19 to age 24 for plot 318027 in a loblolly pine stand (Table 3). The current stand table was given in column (2), and the future stand attributes were $N_2 = 1952$ trees/ha, $\bar{D}_2 = 14.13$ cm and $B_2 = 34.68$ m²/ha (bottom of column 3).

The coefficient b_1 of Equation (1) was calculated using the secant method (Press et al. 1996) to be -0.1155 , so that the stand density after mortality was reduced from 3603 to 1952 trees/ha (column 4). The average diameter and quadratic mean diameter after mortality were 10.95 and 11.98 cm, respectively. The increase in minimum diameter was calculated from equations (5, 6, and 7) to be 2.64 cm. As a result, the minimum diameter increased by one 2 cm class, and the entire diameter distribution was shifted forward by one class (column 5). At this point, the current minimum diameter class was 4 cm, and the average diameter and stand basal area were computed to be 12.95 cm and 29.34 m²/ha, respectively.

The Weibull parameters were recovered from current and future stand attributes as follows:

Time	\bar{D} (cm)	B (m ² /ha)	a	b	c
1	12.95	29.34	3.0	11.2318	2.1508
2	14.13	34.68	3.0	12.5640	2.2885

The diameter growth function implied from the above current and future Weibull parameters was

$$X_2 = 3.0 + 12.5640 \left(\frac{X_1 - 3.0}{11.2318} \right)^{\frac{2.1508}{2.2885}} \quad (17)$$

where X_1 and X_2 are current and future tree diameters, respectively. The current diameter can be solved backwards from the future diameter:

$$X_1 = 3.0 + 11.2318 \left(\frac{X_2 - 3.0}{12.5640} \right)^{\frac{2.2885}{2.1508}} \quad (18)$$

The following example illustrates the calculations of number of trees per hectare for the 6 cm future diameter class that spans from 5 to 7 cm. These two future diameters correspond to current diameters of 4.59 and 6.32 cm, respectively, from Equation (18). The interval [4.59 cm, 6.32 cm] can be divided into two subintervals. The first one extends from 4.59 to 5.00 cm, or 38.70% of the 24.26 trees/ha in the current 4 cm class. The second subinterval covers from 5.00 to 6.32 cm, or 58.59% of the 209.86 trees/ha in the current 6 cm class. The above proportions were computed using Equation (10). Number of tree/ha in the future 6 cm class is therefore:

$$0.3870 (24.26) + 0.5859 (209.86) = 132.34 \text{ trees/ha}$$

Number of trees per hectare for other future diameter classes were determined in the same manner (column 6 of Table 3). Note that the resulting stand basal area was 34.70 m²/ha, which needed to be adjusted to the given future basal area of 34.68 m²/ha.

In the adjustment step, the solution to the system of Equations (16a-c) was $\lambda_1 = 1.3679$, $\lambda_2 = -0.2144$, and $\lambda_3 = 0.0066$. The final stand table (column 7) was computed from Equation (14). This stand table produced stand attributes that matched the specified values of N_2 , \bar{D}_2 , and B_2 .

Figure 1 presents the observed stand tables of plot 307033 at ages 30 and 34, and the projected stand tables at age 34 by the constrained LS method and by the N&S method. Note that both methods had difficulties in handling the first diameter class (8 cm). The increase in arithmetic and quadratic mean diameters was not sufficient in this case for the constrained LS method to detect the increase in minimum diameter. Both methods did well in approximating the "gap" in the 32–36 cm

Table 3. A numerical example demonstrating the application of the new method to projecting stand table from age 19 to age 24 for plot 318027.

Diameter (cm)	Observed trees/ha		Predicted trees/ha			
	Age 19 (2)	Age 24 (3)	After mortality (4)	Shift 1 class (5)	After D growth (6)	Final (7)
2	222.39		24.26			
4	716.58	24.71	209.86	24.26	14.87	14.26
6	741.29	172.97	325.23	209.86	132.34	132.02
8	345.94	148.26	191.83	325.23	256.05	255.98
10	296.52	222.39	191.67	191.83	244.42	244.53
12	444.77	247.10	319.95	191.67	181.96	182.22
14	370.64	345.94	288.08	319.95	230.35	230.69
16	172.97	222.39	142.39	288.08	297.87	298.24
18	197.68	197.68	169.94	142.39	235.78	236.13
20	74.13	148.26	65.87	169.94	146.67	146.95
22	0.00	148.26	0.00	65.87	139.97	140.12
24	19.71	49.42	22.98	0.00	48.79	48.76
26		0.00		22.98	5.57	5.30
28		24.71			17.41	16.86
N^*	3,603	1,952	1,952	1,952	1,952	1,952
\bar{D}	8.97	14.13	10.95	12.95	14.13	14.13
Q	10.23	15.04	11.98	13.83	15.04	15.04
B	29.60	34.68	22.01	29.34	34.70	34.68

* N is total number of trees / ha, \bar{D} is average diameter in cm, Q is quadratic mean diameter in cm, and B is basal area in m²/ha.

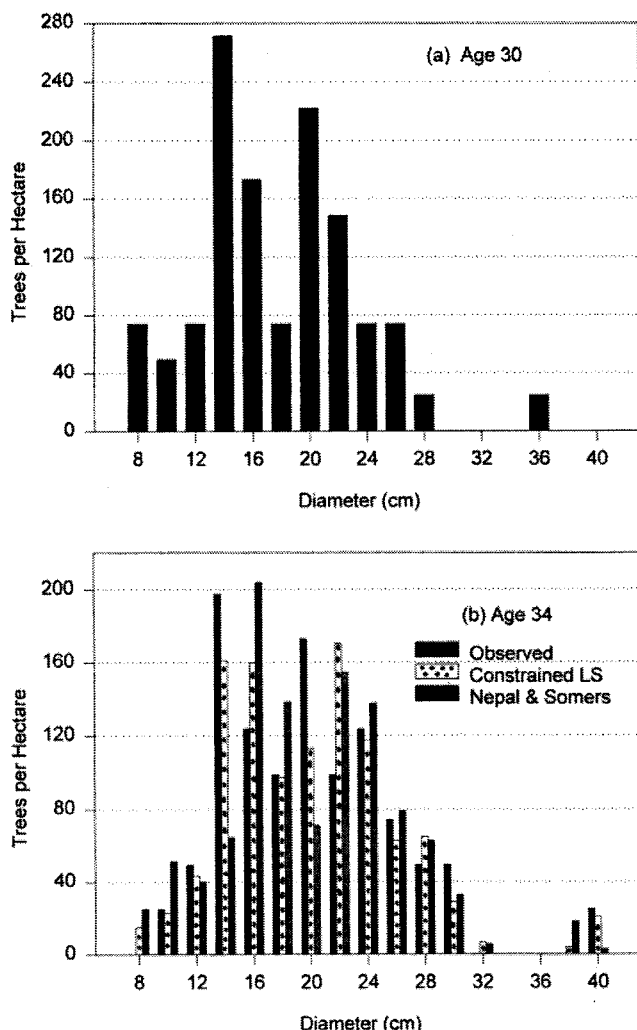


Figure 1. Observed diameter distributions for plot 307033 at (a) age 30 and (b) age 34. Diagram (b) also shows predicted diameter distributions by the constrained least squares method and Nepal and Somers' (1992) model.

classes. The observed 25 trees/ha at the 40 cm diameter class created problems for the N&S method, which predicted 18 trees/ha at the 38 cm class and only 3 trees at the 40 cm class. The constrained LS method performed better by predicting 4 and 20 trees/ha at the 38 and 40 cm classes, respectively.

Evaluation Criteria

In this study, the constrained LS model was evaluated against the N&S method and the Weibull parameter-recovery method, using the data from direct-seeded stands of loblolly pines. The evaluation criteria included the two-sample Kolmogorov-Smirnov (K-S) statistic (Steel and Torrie 1980), the chi-square (χ^2) statistic (Steel and Torrie 1980), and an

error index similar to that proposed by Reynolds et al. (1988). This error index was computed for each plot as the sum of absolute differences between observed and predicted proportions of trees in each diameter class. The three statistics measured the goodness-of-fit of the predicted future stand tables when compared to the observed stand tables, with lower values indicating a better fit.

Results and Discussion

Means and standard deviations of the evaluation statistics for each of the three methods are presented in Table 4. The Weibull parameter-recovery method consistently ranked in third place based on all statistics. This suggests that some plots might have irregular or multimodal diameter distributions that could not be well approximated by a single Weibull distribution.

The constrained LS method resulted in the lowest mean values of all evaluation statistics. The reduction in values of the statistics from the N&S method to the constrained LS approach was 10% for the K-S statistic, 20% for the χ^2 statistic, and 7% for the error index. Duncan's multiple range tests showed that the three methods were significantly different at the 5% level for all statistics (Table 4). The K-S tests failed to reject the null hypothesis that the observed and predicted stand tables came from the same population at the 5% level for all three methods. The χ^2 tests, performed at the 5% level, rejected that hypothesis on 9.11%, 9.30%, and 4.17% of the stand tables for the Weibull parameter-recovery approach, the N&S method, and the constrained LS method, respectively. These results indicate that on the average, the constrained LS method produced stand tables that better approximated observed future stand tables than did the N&S method.

There are three possible explanations for the better performance of the constrained LS method.

1. Nepal and Somers (1992) derived the implied diameter growth equation based on future stand attributes and current stand attributes *before mortality*. The proposed method used current stand attributes *after mortality* for this purpose; therefore, it improved predictions of diameter increment by filtering out the confounding effect of tree mortality. The diameter growth function [Equation (9)] was recovered from current and future stand attributes. It makes good sense to include in these calculations only trees that would survive the growing period.
2. The adjustment step in the constrained LS method made sure that the resulting stand tables produced the following compatible stand attributes: trees and basal area per

Table 4. Means (and standard deviations) of the evaluation statistics by method ($n = 527$).

Method	Evaluation statistic		
	K-S	χ^2	Error index
Weibull	0.058a (0.028)*	9.063a (5.821)	0.286a (0.135)
Nepal and Somers	0.050b (0.026)	8.002b (6.428)	0.261b (0.131)
Constrained LS	0.045c (0.023)	6.389c (4.804)	0.243c (0.121)

* For each evaluation statistic, means with the same letter are not significantly different at the 5% level (from the Duncan's multiple range test).

hectare and average diameter. The latter attribute was not employed as a constraint in the N&S method; its addition to the list of constraints incorporated in the new method resulted in better stand table projections.

3. The minimum diameter is not allowed to change in the N&S model but will increase in the constrained LS method in cases where there is a sufficient shift in the diameter distribution. As a result, the constrained LS model provided better projections of the stand tables in these cases.

The constraints that the predicted stand tables must match future stand attributes are logical when the stand attributes are observed values. In this case, the constraints ensure that predicted stand tables are closer to the observed stand tables. On the other hand, stand tables may be derived from future stand attributes predicted from a stand-level growth and yield model. The reliability of these stand tables depends heavily on how well the stand-level model predicts for the population.

Conclusion

The proposed constrained LS method consists of three logical steps: (1) predict survival in each diameter class, (2) predict diameter growth from the current and future stand attributes, and (3) adjust the stand table to match the future average diameter and stand basal area. Results from evaluations against the Weibull parameter-recovery approach suggest that the constrained LS method can perform well on diameter data that do not exactly follow the Weibull distribution.

In addition to the constraints of number of trees and basal area per hectare, the extra constraint of average diameter ensured that predicted stand tables more closely

approximate observed stand tables. That explained the overall better performance of the constrained LS method compared to Nepal and Somers's (1992) method on the direct-seeded loblolly pine data set.

Literature Cited

- AVERY, T.E., AND H.E. BURKHART. 1994. Forest measurements. Ed. 4. McGraw-Hill, New York. 408 p.
- BAILEY, R.L. 1980. Individual tree growth derived from diameter distribution models. *For. Sci.* 26:626-632.
- BAILEY, R.L., AND T.R. DELL. 1973. Quantifying diameter distributions with the Weibull function. *For. Sci.* 19:97-104.
- CHAPMAN, H.H., AND W.H. MEYER. 1949. Forest mensuration. McGraw-Hill, New York. 522 p.
- CLUTTER, J.L., AND E.P. JONES, JR. 1980. Prediction of growth after thinning in old-field slash pine plantations. USDA For Serv. Res. Pap. SE-217. 14 p.
- MATNEY, T.G., K.L. BELLI, AND R.M. FARRAR, JR. 1990. Parameter-free diameter-distribution recovery. P. 95-107 in Division 4 Proc. IUFRO World Congress meet. Publ. FWS-2-90; School of For. & Wildl. Resour., VPI&SU, Blacksburg, VA.
- NEPAL, S.K., AND G.L. SOMERS. 1992. A generalized approach to stand table projection. *For. Sci.* 38:120-133.
- PIENAAR L.V., AND W.M. HARRISON. 1988. A stand table projection approach to yield prediction in unthinned even-aged stands. *For. Sci.* 34:804-808.
- PRESS, W.H., S.A. TEUKOLSKY, W.T. VETTERLING, AND B.P. FLANNERY. 1996. Numerical recipes in Fortran 77—The art of scientific computing. Ed. 2. Cambridge University Press, New York. 933 p.
- REYNOLDS, M.R., JR., T.E. BURK, AND W.C. HUANG. 1988. Goodness-of-fit tests and model selection procedures for diameter distribution models. *For. Sci.* 34:373-399.
- STEEL, R.G.D., AND J.H. TORRIE. 1980. Principles and procedures of statistics: A biometrical approach. Ed. 2. McGraw-Hill, New York. 633 p.
- TANG, S., Y. WANG, L. ZHANG, AND C. H. MENG. 1997. A distribution-independent approach to predicting stand diameter distribution. *For. Sci.* 43:491-500.

